

# Derivation of exact master equation with stochastic description: Models in quantum optics

Haifeng Li and Jiushu Shao\*

*Key Laboratory of Theoretical Computational Photochemistry,  
Ministry of Education, College of Chemistry,  
Beijing Normal University, Beijing 100875, China*

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## Abstract

The methodology of stochastic description for dissipation, a generic scheme to decouple the interaction between two subsystems, is applied to the study of dissipative dynamics in quantum optics. It is shown that the influence of the coupled thermal or vacuum field on the quantum mode can be exactly represented by the induced stochastic fields. The quantum mode thereby satisfies a stochastic differential equation and dissipation effect due to the coupling with the environment is obtained through statistical averaging. Within the framework of stochastic description, it is demonstrated how to derive the master equation for a single optical mode interacting with the bosonic bath. A numerical algorithm for solving the master equation in which the coefficients are determined by a set of integral equations is discussed and a comparison with the known results is displayed. The derivation of the master equation for the spontaneous decay of two-state atoms in the vacuum is also presented.

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\*Electronic address: jiushu@bnu.edu.cn

## I. INTRODUCTION

The best known dissipative dynamics is perhaps the Brownian motion, which has been greatly explored in theory and well understood since the pioneering work of Einstein [1]. Because the traditional Brownian particle is heavy and its surrounding environment or bath is at sufficiently high temperature, the motion of the particle can be accurately described by classical mechanics [2]. When the system of interest is very light, or the temperature of the environment is low, however, classical mechanics is no longer applicable and one has to invoke quantum mechanics. Actually, all physical systems intrinsically follow quantum mechanics and the traditional theory of Brownian motion should be a classical approximation of the *exact* quantum version. There have been many efforts made to establish a quantum formulation of dissipative dynamics and the most successful one is based on the system plus bath model [3]. Compared to classical counterpart, the quantum evolution exhibits a unique feature, that is, coherence. Quantum coherence is a consequence of the principle of linear superposition and plays indispensable role in the operation of quantum information devices [4]. It is also a fundamental issue related to quantum measurement [5–7]. The main purpose of studies on quantum dissipation is thus to reveal how the environment affects the time evolution of the quantum system, in particular, the decoherence effect [8].

The classical Brownian motion is generally described by a generalized Langevin equation in which the random force is induced by the thermal bath [2, 3]. One naturally wonders whether the impact of the bath can be defined as a *classical* random field. Kubo was the first to propose the stochastic Liouville equation for quantum dissipative systems, but his formulation is phenomenological [9]. Given the system plus bath model, the key issue is then to *acquire* the equation of motion of the system, in which the dissipation effect due to the bath is exactly taken into account and the explicit dynamics of the bath does not show up. In other words, one aims at finding the effective motion of the system in its own physical space instead of inspecting the every detail of the dynamics for the whole system. To this end, several theoretical frameworks including the projection operator technique [10, 11], the influence functional method [12–14], and quantum Langevin equation approach [15–17] were suggested and applied to a wide range of physical systems. Of course, all of these methods have their own pros and cons in practice. We have recently put forward a stochastic theory for dissipative systems, in which the interaction with the bath is rigorously mapped into

stochastic fields during the evolution of the system [18]. In this formulation the system is subjected to complex stochastic fields induced by the bath [18]. For comparison, in Kubo's stochastic Liouville equation, there is only a real stochastic field due to the bath. For specific dissipation, we have proposed the hierarchical equation of motion approach in terms of the stochastic formulation [19, 20], which has become an efficient, deterministic numerical technique of many applications [21]. Combining the stochastic and the deterministic methods, we were able to solve the dynamics of the two-state system strongly coupled to a bosonic bath [22]. Besides the trophy in numerical simulations, it has been shown that the stochastic formulation offers a convenient, systematic procedure for theoretical analysis, say the derivation of the master equation for linear systems [23]. When the existence of master equation is warranted, its derivation and solution should be the central task in quantum dissipative dynamics. This paper, as a continuation of the previous one [23], addresses the dissipative dynamics in quantum optics. We will apply the stochastic formulation to derive the well-known master equations for two models. One is a single mode perturbed by a thermal bath [24] and the other is the spontaneous decay: the two-state atom in the vacuum [18, 25–30].

As discussed in literature, the fluctuations of the thermal field are a major source of dissipation in quantum optics, which damage coherence of the system [31]. Again, because quantum optical devices operate when decoherence is negligible, to realize optimal functioning, it is sometimes necessary to design a scheme to control dissipation [26, 31–33]. This entails a clear revelation of underlying dissipative dynamics. It is no doubt that exactly solvable models may provide fundamental understanding in this respect and are always desired. In the previous paper we have shown how to employ the stochastic description of quantum dissipation to work out the master equation [23]. The harmonic oscillator coupled to the Caldeira-Leggett heat bath is used as an example. In this paper the thermal field as well as the vacuum is considered to be the heat bath. There are two kinds of interaction between the system and the bath, one corresponding to the absorption and the other the emission, of a photon energy [24, 34–36]. Although these models can formally be regarded as a result of rotating-wave approximation (RWA) imposed on the Caldeira-Leggett model, we will avoid the discussion on the validity of the approximation [37].

The first model we shall consider is essentially the dissipative harmonic oscillator within RWA. Its master equation was derived by Xiong *et al.* [24], resorting to the influence func-

tional approach developed by Feynman and Vernon [12, 13]. As the authors observed, when the coupling between the system and the bath becomes strong, the dissipative dynamics will change dramatically because of the non-Markovian memory effect [24]. In a recent paper [38], Tan and Zhang used the same method to discuss the consequence of initial system-bath correlation. The second model describes the spontaneous decay of two-state atoms in vacuum, which is exactly solvable. It has been frequently used as benchmark calculations in quantum optics. Indeed, this model is so well-known that diversified methods have been used to derive its master equation [18, 25–30].

The paper is organized as follows. In Sec. II we recapitulate the stochastic formulation for quantum dissipation. In Sec. III we apply the scheme to a single-mode cavity system coupled to a thermal field and derive its exact master equation. In Sec. IV the obtained master equation is shown to be equivalent to the result in Ref. [24] and some remarks on the numerical implementation are given. In Sec. V the master equation of the same system subjected to a driving external field is derived. In Sec. VI we show how to derive the master equation of a two-state atom coupled to the vacuum field. We present our conclusions in Sec. VII.

## II. THEORY

To study the dissipative dynamics of a quantum mode in an optical cavity, we start with an arbitrary cavity system coupled to a thermal field consisting of infinite number of harmonic oscillators. The Hamiltonian of the entire system assumes

$$\hat{H} = \hat{H}_s + \sum_j \hbar \omega_j b_j^\dagger b_j + \sum_j \hbar \left( c_j \hat{f}_1 b_j^\dagger + c_j \hat{f}_2 b_j \right), \quad (1)$$

where the first term on the right-hand side is the Hamiltonian of the cavity mode, the second term is the Hamiltonian of the thermal bath, and the last two terms define the interaction between the system and the bath. Here  $\hat{f}_1$  and  $\hat{f}_2$  are operators for the system and they are a hermitian pair,  $\hat{f}_1 = \hat{f}_2^\dagger$ . Note that the two interaction terms can be interpreted as emitting and absorbing a quantum phonon or photon by the bath. The model will be the Caldeira-Leggett type when the interaction is of the form  $\sum_j \hbar c_j \left( \hat{f}_1 + \hat{f}_2 \right) \left( b_j^\dagger + b_j \right)$ . As shown in the previous papers [18], the dissipative dynamics can be described by a stochastic formulation in which the system evolves in the stochastic fields induced by the bath and the

statistical average of the random density matrix is nothing but the reduced density matrix. For the model we consider, the random density matrix satisfies

$$i\hbar d\rho_s(t) = \left[ \hat{H}_s + \sum_{k=1}^2 \bar{g}_k(t) \hat{f}_k, \rho_s \right] dt + \frac{\sqrt{\hbar}}{2} \sum_{k=1}^2 [\hat{f}_k, \rho_s] dW_{1k} + i \frac{\sqrt{\hbar}}{2} \sum_{k=1}^2 \{ \hat{f}_k, \rho_s \} dW_{2k}^*, \quad (2)$$

where the bath-induced stochastic fields are given by

$$\bar{g}_1(t) = \sum_j \frac{\text{Tr}_b \{ \hbar c_j b_j^\dagger \rho_b(t) \}}{\text{Tr}_b \{ \rho_b(t) \}} \quad (3)$$

and

$$\bar{g}_2(t) = \sum_j \frac{\text{Tr}_b \{ \hbar c_j b_j \rho_b(t) \}}{\text{Tr}_b \{ \rho_b(t) \}}. \quad (4)$$

Here, introduced are the complex Wiener processes  $W_{1k}(t) = \int_0^t dt' [\nu_{1k}(t') + i\nu_{4k}(t')]$  and  $W_{2k}(t) = \int_0^t dt' [\nu_{2k}(t') + i\nu_{3k}(t')]$ , where  $\nu_{nk}(t)$  ( $n = 1 - 4$ ) are independent Gaussian white noises with zero mean and delta function correlation. This is the main result of the stochastic formulation and will be the working formula. To use it, of course, we need to first calculate  $\bar{g}_1(t)$  and  $\bar{g}_2(t)$ . In this formulation,  $\bar{g}_1(t)$  and  $\bar{g}_2(t)$  can be determined by the evolution of the bath,

$$\begin{aligned} i\hbar d\rho_b = & \sum_j \left[ \hbar\omega_j b_j^\dagger b_j, \rho_b \right] dt + \frac{\hbar\sqrt{\hbar}}{2} \sum_j c_j [b_j^\dagger, \rho_b] dW_{21} + \frac{\hbar\sqrt{\hbar}}{2} \sum_j c_j [b_j, \rho_b] dW_{22} \\ & + i \frac{\hbar\sqrt{\hbar}}{2} \sum_j c_j \{ b_j^\dagger, \rho_b \} dW_{11}^* + i \frac{\hbar\sqrt{\hbar}}{2} \sum_j c_j \{ b_j, \rho_b \} dW_{12}^*. \end{aligned} \quad (5)$$

The formal solution of  $\rho_b(t)$  can be written as

$$\rho_b(t) = u_1(t, 0) \rho_b(0) u_2(0, t), \quad (6)$$

where  $u_{1,2}(t, 0)$  are the forward and backward propagators dictated by

$$\hat{h}_1(t) = \sum_j \hbar\omega_j b_j^\dagger b_j + \frac{\hbar\sqrt{\hbar}}{2} \sum_j c_j b_j^\dagger \eta_{11}(t) + \frac{\hbar\sqrt{\hbar}}{2} \sum_j c_j b_j \eta_{12}(t)$$

and

$$\hat{h}_2(t) = \sum_j \hbar\omega_j b_j^\dagger b_j + \frac{\hbar\sqrt{\hbar}}{2} \sum_j c_j b_j^\dagger \eta_{21}(t) + \frac{\hbar\sqrt{\hbar}}{2} \sum_j c_j b_j \eta_{22}(t).$$

with

$$\begin{aligned}
\eta_{11}(t) &= \nu_{21}(t) + i\nu_{31}(t) + i\nu_{11}(t) + \nu_{41}(t), \\
\eta_{12}(t) &= \nu_{22}(t) + i\nu_{32}(t) + i\nu_{12}(t) + \nu_{42}(t), \\
\eta_{21}(t) &= \nu_{21}(t) + i\nu_{31}(t) - i\nu_{11}(t) - \nu_{41}(t), \\
\eta_{22}(t) &= \nu_{22}(t) + i\nu_{32}(t) - i\nu_{12}(t) - \nu_{42}(t),
\end{aligned}$$

being complex white noises. Because the bath modes are independent, the propagator of the bath is a product of the individual ones, namely,  $u_1(t, 0) = \prod_j u_{j,1}(t, 0)$  and  $u_2(0, t) = \prod_j u_{j,2}(0, t)$ .

As illustrated in the previous paper and other references [23, 39], the propagator for each bath mode can feasibly be obtained upon using the interaction representation. As a result, the forward propagator  $u_{1,j}(t, 0)$  reads

$$u_{j,1}(t, 0) = C_{j,10}(t) e^{C_{j,11}(t)b_j} e^{C_{j,12}(t)b_j^\dagger} u_{j,0}(t, 0), \quad (7)$$

where  $u_{j,0}(t, 0)$  is the propagator of the undriven harmonic oscillator described by  $h_{j,0} = \hbar\omega_j b_j^\dagger b_j$ , which is well known [12, 13, 40–42], and

$$\begin{aligned}
C_{j,10}(t) &= \exp \left[ \frac{\hbar}{4} c_j^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \eta_{11}(t_1) \eta_{12}(t_2) e^{i\omega_j(t_1-t_2)} \right], \\
C_{j,11}(t) &= -i \frac{\sqrt{\hbar}}{2} c_j \int_0^t dt_1 \eta_{12}(t_1) e^{i\omega_j(t-t_1)}, \\
C_{j,12}(t) &= -i \frac{\sqrt{\hbar}}{2} c_j \int_0^t dt_1 \eta_{11}(t_1) e^{-i\omega_j(t-t_1)}.
\end{aligned}$$

Similarly, the backward propagator  $u_{j,2}(0, t)$  is

$$u_{j,2}(0, t) = C_{j,20}(t) u_{j,0}(0, t) e^{C_{j,22}(t)b_j^\dagger} e^{C_{j,21}(t)b_j}, \quad (8)$$

where

$$\begin{aligned}
C_{j,20}(t) &= \exp \left[ -\frac{\hbar}{4} c_j^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \eta_{21}(t_1) \eta_{22}(t_2) e^{i\omega_j(t_1-t_2)} \right], \\
C_{j,21}(t) &= i \frac{\sqrt{\hbar}}{2} c_j \int_0^t dt_1 \eta_{22}(t_1) e^{i\omega_j(t-t_1)}, \\
C_{j,22}(t) &= i \frac{\sqrt{\hbar}}{2} c_j \int_0^t dt_1 \eta_{21}(t_1) e^{-i\omega_j(t-t_1)}.
\end{aligned}$$

Assume that the thermal field is initially in thermal equilibrium,

$$\rho_b(0) = \frac{1}{Z_b} e^{-\beta \hat{H}_b} = \frac{1}{Z_b} \prod_j e^{-\beta \hbar \omega_j}, \quad (9)$$

where  $Z_b = \text{Tr}_b \{e^{-\beta \hat{H}_b}\}$  is the partition function. Inserting together with Eqs. (7) and (8) into Eq. (6), carrying some operator algebra and rearranging, we obtain

$$\rho_b(t) = \prod_j F_j(t) \exp \left\{ [C_{j,12}(t) + C_{j,22}(t)e^{-\beta \hbar \omega_j}] b_j^\dagger \right\} \exp \left\{ [C_{j,11}(t) + C_{j,21}(t)e^{\beta \hbar \omega_j}] b_j \right\} \exp \left( -\beta \hat{H}_b \right), \quad (10)$$

where

$$F_j(t) = \frac{1}{Z_b} C_{j,10}(t) C_{j,20}(t) \exp \left\{ C_{j,11}(t) [C_{j,12}(t) + C_{j,22}(t)e^{-\beta \hbar \omega_j}] \right\}.$$

Then the bath-induced fields defined by Eqs. (3) and (4) can be worked out,

$$\bar{g}_1(t) = \frac{i\hbar\sqrt{\hbar}}{2} \int_0^t dt' \{ \alpha_1(t-t') [\nu_{22}(t') + i\nu_{32}(t')] - \alpha_2(t-t') [i\nu_{12}(t') + \nu_{42}(t')] \} \quad (11)$$

and

$$\bar{g}_2(t) = -\frac{i\hbar\sqrt{\hbar}}{2} \int_0^t dt' \{ \alpha_1^*(t-t') [\nu_{21}(t') + i\nu_{31}(t')] + \alpha_2^*(t-t') [i\nu_{11}(t') + \nu_{41}(t')] \}, \quad (12)$$

where  $\alpha_1(t)$  and  $\alpha_2(t)$  are response functions given by

$$\alpha_1(t) = \int_0^\infty d\omega J(\omega) e^{i\omega t} \quad (13)$$

and

$$\alpha_2(t) = \int_0^\infty d\omega J(\omega) \coth \left( \frac{\beta}{2} \hbar \omega \right) e^{i\omega t}, \quad (14)$$

$J(\omega)$  being the spectral density function

$$J(\omega) = \sum_j c_j^2 \delta(\omega_j - \omega). \quad (15)$$

We like to stress that  $J(\omega)$  completely captures the influence of the thermal field on the system. With the known  $\bar{g}_{1,2}(t)$ , Eq. (2) is a closed stochastic differential equation. That is, one can generate the required stochastic fields through white noises and solve Eq. (2) for a given initial condition  $\rho_s(0)$ . The reduced density matrix  $\tilde{\rho}_s(t)$  is of course the mathematical expectation of random density matrix  $\rho_s(t)$ , namely,  $\tilde{\rho}_s(t) = M \{ \rho_s(t) \}$ . We can also try to

find the equation of motion for  $\tilde{\rho}_s(t)$  from Eq. (2). To this end, we take stochastic averaging to obtain

$$i\hbar \frac{d\tilde{\rho}_s}{dt} = [\hat{H}_s, \tilde{\rho}_s] + \frac{i\hbar\sqrt{\hbar}}{2} \left[ \hat{f}_1, \int_0^t dt' [\alpha_1(t-t')\hat{O}_{s,11}(t,t') - \alpha_2(t-t')\hat{O}_{s,12}(t,t')] \right] \\ - \frac{i\hbar\sqrt{\hbar}}{2} \left[ \hat{f}_2, \int_0^t dt' [\alpha_1^*(t-t')\hat{O}_{s,21}(t,t') + \alpha_2^*(t-t')\hat{O}_{s,22}(t,t')] \right], \quad (16)$$

where the dissipative operators are

$$\hat{O}_{s,11}(t,t') = M \left\{ \frac{\delta\rho_s(t)}{\delta\nu_{22}(t')} + i \frac{\delta\rho_s(t)}{\delta\nu_{32}(t')} \right\}, \quad (17)$$

$$\hat{O}_{s,12}(t,t') = M \left\{ i \frac{\delta\rho_s(t)}{\delta\nu_{12}(t')} + \frac{\delta\rho_s(t)}{\delta\nu_{42}(t')} \right\}, \quad (18)$$

$$\hat{O}_{s,21}(t,t') = M \left\{ \frac{\delta\rho_s(t)}{\delta\nu_{21}(t')} + i \frac{\delta\rho_s(t)}{\delta\nu_{31}(t')} \right\}, \quad (19)$$

$$\hat{O}_{s,22}(t,t') = M \left\{ i \frac{\delta\rho_s(t)}{\delta\nu_{11}(t')} + \frac{\delta\rho_s(t)}{\delta\nu_{41}(t')} \right\}. \quad (20)$$

In the above derivation the nonanticipating property of  $\rho_s(t)$ , namely,  $M\{\rho_s(t)dW_{m,k}(t)\} = 0$  ( $m = 1, 2$ ), and the Furutsu-Novikov theorem [43], that is,  $M\{\nu(t')F[\nu]\} = M\{\delta F[\nu]/\delta\nu(t')\}$  for a white noise  $\nu(t)$  and its arbitrary functional  $F[\nu]$ , are used. As shown in Ref. [18], the formal solution of Liouville-like Eq. (2) can be written as

$$\rho_s(t) = U_1(t, 0)\rho_s(0)U_2(0, t), \quad (21)$$

where  $U_1(t, 0)$  is the forward propagator associated with the Hamiltonian

$$\hat{H}_1(t) = \hat{H}_s + \xi_{11}(t)\hat{f}_1 + \xi_{12}(t)\hat{f}_2 \quad (22)$$

while  $U_2(0, t)$  is the backward propagator associated with the Hamiltonian

$$\hat{H}_2(t) = \hat{H}_s + \xi_{21}(t)\hat{f}_1 + \xi_{22}(t)\hat{f}_2 \quad (23)$$

with

$$\xi_{11}(t) = \bar{g}_1(t) + i \frac{\sqrt{\hbar}}{2} \eta_{11}^*(t), \\ \xi_{12}(t) = \bar{g}_2(t) + i \frac{\sqrt{\hbar}}{2} \eta_{12}^*(t), \\ \xi_{21}(t) = \bar{g}_1(t) - i \frac{\sqrt{\hbar}}{2} \eta_{21}^*(t), \\ \xi_{22}(t) = \bar{g}_2(t) - i \frac{\sqrt{\hbar}}{2} \eta_{22}^*(t).$$

Following Ref [18], we calculate the functional derivatives to obtain *formal* solutions of the dissipative operators,

$$\hat{O}_{s,11}(t, t') = \frac{1}{\sqrt{\hbar}} M \left\{ U_1(t, t') \hat{f}_2 U_1(t', t) \rho_s(t) + \rho_s(t) U_2(t, t') \hat{f}_2 U_2(t', t) \right\}, \quad (24)$$

$$\hat{O}_{s,12}(t, t') = \frac{1}{\sqrt{\hbar}} M \left\{ U_1(t, t') \hat{f}_2 U_1(t', t) \rho_s(t) - \rho_s(t) U_2(t, t') \hat{f}_2 U_2(t', t) \right\}, \quad (25)$$

$$\hat{O}_{s,21}(t, t') = \frac{1}{\sqrt{\hbar}} M \left\{ U_1(t, t') \hat{f}_1 U_1(t', t) \rho_s(t) + \rho_s(t) U_2(t, t') \hat{f}_1 U_2(t', t) \right\}, \quad (26)$$

$$\hat{O}_{s,22}(t, t') = \frac{1}{\sqrt{\hbar}} M \left\{ U_1(t, t') \hat{f}_1 U_1(t', t) \rho_s(t) - \rho_s(t) U_2(t, t') \hat{f}_1 U_2(t', t) \right\}. \quad (27)$$

When these operators can be expressed in terms of the reduced density operator  $\tilde{\rho}_s(t)$  and other operators of the system, then Eq. (16) becomes a master equation. In the following section, we show that a dissipative single-mode optical cavity is indeed the case.

### III. MASTER EQUATION FOR DISSIPATIVE SINGLE-MODE OPTICAL CAVITY

Let us consider a single-mode cavity,  $\hat{H}_s = \hbar\omega_0 a^\dagger a$ , with the coupling operators  $\hat{f}_1 = a$  and  $\hat{f}_2 = a^\dagger$ . Therefore, the dynamics of the random cavity are determined by the forward and backward Hamiltonians

$$\hat{H}_1(t) = \hbar\omega_0 a^\dagger a + \xi_{11}(t)a + \xi_{12}(t)a^\dagger \quad (28)$$

and

$$\hat{H}_2(t) = \hbar\omega_0 a^\dagger a + \xi_{21}(t)a + \xi_{22}(t)a^\dagger. \quad (29)$$

These Hamiltonians are exactly solvable and one readily finds

$$\begin{aligned} U_1(t, t') a U_1(t', t) &= a e^{i\omega_0(t-t')} + \frac{i}{\hbar} \int_{t'}^t dt_1 \xi_{12}(t_1) e^{i\omega_0(t_1-t')}, \\ U_2(t, t') a U_2(t', t) &= a e^{i\omega_0(t-t')} + \frac{i}{\hbar} \int_{t'}^t dt_1 \xi_{22}(t_1) e^{i\omega_0(t_1-t')}, \\ U_1(t, t') a^\dagger U_1(t', t) &= a^\dagger e^{-i\omega_0(t-t')} - \frac{i}{\hbar} \int_{t'}^t dt_1 \xi_{11}(t_1) e^{-i\omega_0(t_1-t')}, \\ U_2(t, t') a^\dagger U_2(t', t) &= a^\dagger e^{-i\omega_0(t-t')} - \frac{i}{\hbar} \int_{t'}^t dt_1 \xi_{21}(t_1) e^{-i\omega_0(t_1-t')}. \end{aligned}$$

Inserting into Eqs. (24)–(27) and taking statistical averaging, we obtain

$$\begin{aligned}\hat{O}_{s,11}(t, t') &= \frac{1}{\sqrt{\hbar}} \{a^\dagger, \tilde{\rho}_s(t)\} e^{-i\omega_0(t-t')} + \int_{t'}^t dt_1 \int_0^{t_1} dt_2 e^{-i\omega_0(t_1-t')} \alpha_1(t_1 - t_2) \hat{O}_{s,11}(t, t_2) \\ &\quad - \int_{t'}^t dt_1 \int_0^{t_1} dt_2 e^{-i\omega_0(t_1-t')} \alpha_2(t_1 - t_2) \hat{O}_{s,12}(t, t_2),\end{aligned}\quad (30)$$

$$\hat{O}_{s,12}(t, t') = \frac{1}{\sqrt{\hbar}} [a^\dagger, \tilde{\rho}_s(t)] e^{-i\omega_0(t-t')} - \int_{t'}^t dt_1 \int_{t_1}^t dt_2 e^{-i\omega_0(t_1-t')} \alpha_1^*(t_2 - t_1) \hat{O}_{s,12}(t, t_2), \quad (31)$$

$$\begin{aligned}\hat{O}_{s,21}(t, t') &= \frac{1}{\sqrt{\hbar}} \{a, \tilde{\rho}_s(t)\} e^{i\omega_0(t-t')} + \int_{t'}^t dt_1 \int_0^{t_1} dt_2 e^{i\omega_0(t_1-t')} \alpha_1^*(t_1 - t_2) \hat{O}_{s,21}(t, t_2) \\ &\quad + \int_{t'}^t dt_1 \int_0^{t_1} dt_2 e^{i\omega_0(t_1-t')} \alpha_2(t_2 - t_1) \hat{O}_{s,22}(t, t_2),\end{aligned}\quad (32)$$

$$\hat{O}_{s,22}(t, t') = \frac{1}{\sqrt{\hbar}} [a, \tilde{\rho}_s(t)] e^{i\omega_0(t-t')} - \int_{t'}^t dt_1 \int_{t_1}^t dt_2 e^{i\omega_0(t_1-t')} \alpha_1(t_2 - t_1) \hat{O}_{s,22}(t, t_2). \quad (33)$$

In the above derivation, the following functional derivatives with respective white noises are used,

$$\begin{aligned}M \left\{ \frac{\delta \rho_s(t)}{\delta \nu_{11}(t')} + i \frac{\delta \rho_s(t)}{\delta \nu_{41}(t')} \right\} &= -i\hbar \int_{t'}^t dt_1 \alpha_2^*(t_1 - t') \hat{O}_{s,12}(t, t_1), \\ M \left\{ i \frac{\delta \rho_s(t)}{\delta \nu_{21}(t')} + \frac{\delta \rho_s(t)}{\delta \nu_{31}(t')} \right\} &= -i\hbar \int_{t'}^t dt_1 \alpha_1^*(t_1 - t') \hat{O}_{s,12}(t, t_1), \\ M \left\{ \frac{\delta \rho_s(t)}{\delta \nu_{12}(t')} + i \frac{\delta \rho_s(t)}{\delta \nu_{42}(t')} \right\} &= -i\hbar \int_{t'}^t dt_1 \alpha_2(t_1 - t') \hat{O}_{s,22}(t, t_1), \\ M \left\{ i \frac{\delta \rho_s(t)}{\delta \nu_{22}(t')} + \frac{\delta \rho_s(t)}{\delta \nu_{32}(t')} \right\} &= i\hbar \int_{t'}^t dt_1 \alpha_1(t_1 - t') \hat{O}_{s,22}(t, t_1),\end{aligned}$$

which can be found directly through the solution of  $\rho_s(t)$ . Note that  $\hat{O}_{s,11}^\dagger(t, t') = \hat{O}_{s,21}(t, t')$  and  $\hat{O}_{s,12}^\dagger(t, t') = -\hat{O}_{s,22}(t, t')$ . By iteration, one can show that the dissipative operators should assume the following forms,

$$\hat{O}_{s,11}(t, t') = x_{11}(t, t') \{a^\dagger, \tilde{\rho}_s(t)\} + x_{12}(t, t') [a^\dagger, \tilde{\rho}_s(t)], \quad (34)$$

$$\hat{O}_{s,12}(t, t') = x_{21}(t, t') [a^\dagger, \tilde{\rho}_s(t)], \quad (35)$$

$$\hat{O}_{s,21}(t, t') = x_{11}^*(t, t') \{a, \tilde{\rho}_s(t)\} - x_{12}^*(t, t') [a, \tilde{\rho}_s(t)], \quad (36)$$

$$\hat{O}_{s,22}(t, t') = x_{21}^*(t, t') [a, \tilde{\rho}_s(t)]. \quad (37)$$

Because the operators  $\{a^\dagger, \tilde{\rho}_s(t)\}$ ,  $[a^\dagger, \tilde{\rho}_s(t)]$ ,  $\{a, \tilde{\rho}_s(t)\}$ , and  $[a, \tilde{\rho}_s(t)]$  are arbitrary, it is straightforward to find out the equation of motion for  $x_{mk}(t, t')$  ( $m, k = 1, 2$ ) with

Eqs. (30)–(33). The results are

$$x_{11}(t, t') = \frac{1}{\sqrt{\hbar}} e^{-i\omega_0(t-t')} + \int_{t'}^t dt_1 \int_0^{t_1} dt_2 e^{-i\omega_0(t_1-t')} \alpha_1(t_1 - t_2) x_{11}(t, t_2), \quad (38)$$

$$\begin{aligned} x_{12}(t, t') &= \int_{t'}^t dt_1 \int_0^{t_1} dt_2 e^{-i\omega_0(t_1-t')} \alpha_1(t_1 - t_2) x_{12}(t, t_2) \\ &\quad - \int_{t'}^t dt_1 \int_0^{t_1} dt_2 e^{-i\omega_0(t_1-t')} \alpha_2(t_1 - t_2) x_{21}(t, t_2), \end{aligned} \quad (39)$$

$$x_{21}(t, t') = \frac{1}{\sqrt{\hbar}} e^{-i\omega_0(t-t')} - \int_{t'}^t dt_1 \int_{t_1}^t dt_2 e^{-i\omega_0(t_1-t')} \alpha_1^*(t_2 - t_1) x_{21}(t, t_2). \quad (40)$$

Whenever these coefficients are solved, the dissipative operators Eqs. (34)–(37) become available. Inserting into Eq. (16), we obtain the master equation,

$$\begin{aligned} \frac{d\tilde{\rho}_s(t)}{dt} &= -iA_1(t) [a^\dagger a, \tilde{\rho}_s(t)] + A_2(t) [2a\tilde{\rho}_s(t)a^\dagger - a^\dagger a\tilde{\rho}_s(t) - \tilde{\rho}_s(t)a^\dagger a] \\ &\quad + A_3(t) [a^\dagger \tilde{\rho}_s(t)a + a\tilde{\rho}_s(t)a^\dagger - a^\dagger a\tilde{\rho}_s(t) - \tilde{\rho}_s(t)aa^\dagger], \end{aligned} \quad (41)$$

where the coefficients  $A_j(t)$  ( $j = 1 - 3$ ) are defined by

$$A_1(t) = \omega_0 + \text{Im} \left[ \sqrt{\hbar} \int_0^t dt' \alpha_1^*(t - t') x_{11}^*(t, t') \right], \quad (42)$$

$$A_2(t) = \text{Re} \left[ \sqrt{\hbar} \int_0^t dt' \alpha_1^*(t - t') x_{11}^*(t, t') \right], \quad (43)$$

$$A_3(t) = \text{Re} \left[ \sqrt{\hbar} \int_0^t dt' \alpha_2^*(t - t') x_{21}^*(t, t') \right] - \text{Re} \left[ \sqrt{\hbar} \int_0^t dt' \alpha_1^*(t - t') y^*(t, t') \right], \quad (44)$$

with

$$y^*(t, t') = x_{11}^*(t, t') + x_{12}^*(t, t'). \quad (45)$$

It is clear that  $A_1(t)$  is a frequency-renormalization coefficient,  $A_2(t)$  and  $A_3(t)$  are related to the conventional dissipation and fluctuation coefficients, respectively. In the following section, we will show the equivalence between our derived master equation and that by Xiong *et al.* in terms of path integral approach [24].

#### IV. COMPARISON WITH KNOWN RESULTS

Resorting to the influence functional method developed by Feynman and Vernon, Xiong *et al.* elaborated the derivation of the master equation of optical cavity coupled to a heat bath [24]. For the case of a dissipative single-mode, their result is of the same form as

Eq. (41) and the corresponding coefficients read

$$B_1(t) = \omega_0 + \text{Im} \left[ \sqrt{\hbar} \int_0^t dt' \alpha_1^*(t-t') \bar{x}_{11}(t, t') \right], \quad (46)$$

$$B_2(t) = \text{Re} \left[ \sqrt{\hbar} \int_0^t dt' \alpha_1^*(t-t') \bar{x}_{11}(t, t') \right], \quad (47)$$

$$B_3(t) = \text{Re} \left[ \sqrt{\hbar} \int_0^t dt' \alpha_2^*(t-t') \bar{x}_{21}(t, t') \right] - \text{Re} \left[ \sqrt{\hbar} \int_0^t dt' \alpha_1^*(t-t') \bar{y}(t, t') \right]. \quad (48)$$

The functions  $\bar{x}_{11}(t, t')$ ,  $\bar{x}_{21}(t, t')$ , and  $\bar{y}(t, t')$  are defined by

$$\bar{x}_{11}(t, t') = \frac{1}{\sqrt{\hbar}} u(t') u^{-1}(t), \quad (49)$$

$$\bar{x}_{21}(t, t') = \frac{1}{\sqrt{\hbar}} u^*(t-t'), \quad (50)$$

$$\bar{y}(t, t') = \frac{1}{\sqrt{\hbar}} \bar{u}^*(t') - \frac{2}{\sqrt{\hbar}} [u(t') u^{-1}(t) v(t) - v(t')], \quad (51)$$

where  $u(t)$  and  $v(t)$  obey the following integro-differential equations,

$$\dot{u}(\tau) + i\omega_0 u(\tau) + \int_0^\tau dt' \alpha_1^*(\tau-t') u(t') = 0 \quad (52)$$

and

$$\dot{v}(\tau) + i\omega_0 v(\tau) + \int_0^\tau dt' \alpha_1^*(\tau-t') v(t') = \frac{1}{2} \int_0^t dt' [\alpha_2^*(\tau-t') - \alpha_1^*(\tau-t')] \bar{u}^*(t') \quad (53)$$

with the initial conditions  $u(0) = 1$ ,  $v(0) = 0$ , and  $\bar{u}(\tau) \equiv u(t - \tau)$ . To prove the equivalence of the results obtained by two different methods we only need to prove that  $A_j(t) = B_j(t)$  ( $j = 1 - 3$ ), respectively. As displayed in Eqs. (42)–(44) and Eqs. (46)–(48), all definite integrals in the functions  $A_j(t)$  and  $B_j(t)$  are taken over the same time range  $[0, t]$ . Therefore, a sufficient condition for  $A_j(t) = B_j(t)$  is that the corresponding integrands are identical. Moreover, because these integrands consist of the factors  $\alpha_1(t)$  and  $\alpha_2(t)$  that are dependent on the specificity of the dissipation and can be arbitrary, one can further simplify the problem as a proof of following relations,

$$x_{11}^*(t, t') = \bar{x}_{11}(t, t'), \quad (54)$$

$$x_{21}^*(t, t') = \bar{x}_{21}(t, t'), \quad (55)$$

$$y^*(t, t') = \bar{y}(t, t'). \quad (56)$$

**A. Proof of  $A_1(t) = B_1(t)$ ,  $A_2(t) = B_2(t)$**

As clarified above, if Eq. (54) holds, then  $A_1(t) = B_1(t)$ ,  $A_2(t) = B_2(t)$ . Note that  $u(t)$  satisfies the linear integro-differential equation (52) and that  $\bar{x}_{11}(t, t') = u(t')/(\sqrt{\hbar}u(t))$ . When the first argument  $t$  is fixed,  $\bar{x}_{11}(t, t')$  can be seen as a function of the time variable  $t'$ , which also obeys Eq. (52), namely,

$$\frac{\partial}{\partial t'} \bar{x}_{11}(t, t') + i\omega_0 \bar{x}_{11}(t, t') + \int_0^{t'} dt_1 \alpha_1^*(t' - t_1) \bar{x}(t, t_1) = 0. \quad (57)$$

Return to the integral equation of  $x_{11}(t, t')$ , Eq. (38). Calculating the time derivative with respect to  $t'$  and taking the operation of complex conjugation on both sides of Eq. (38), one obtains for  $x_{11}^*(t, t')$  the same equation as Eq. (57). Also, the initial condition for these equations are the same, namely,  $x_{11}^*(t, t')|_{t'=t} = \bar{x}_{11}(t, t')|_{t'=t} = 1/\sqrt{\hbar}$ . Therefore,  $A_1(t) = B_1(t)$  and  $A_2(t) = B_2(t)$  are proved.

**B. Proof of  $A_3(t) = B_3(t)$**

One only needs to demonstrate that Eqs. (55) and (56) hold. A straightforward algebra shows that  $x_{21}(t, t')$  is time-translation invariant, i.e.,  $x_{21}(t, t') = x_{21}(t + \lambda, t' + \lambda)$ , where  $\lambda$  is a constant. It means that  $x_{21}$  is a function of the time difference  $t - t'$ ,  $x_{21}(t, t') = x_{21}(t - t')$ . As a result, Eq. (40) can be simplified as

$$x_{21}(s) = \frac{1}{\sqrt{\hbar}} e^{-i\omega_0 s} - \int_0^s dt_1 \int_{t_1}^s dt_2 e^{-i\omega_0 t_1} \alpha_1^*(t_2 - t_1) x_{21}(s - t_2). \quad (58)$$

Taking the first-order derivation with respect to the argument  $s$  and the complex conjugation, one obtains

$$\frac{d}{ds} x_{21}^*(s) = i\omega_0 x_{21}^*(s) - \int_0^s dt_1 \alpha_1^*(t_1 - s) x_{21}^*(t_1), \quad (59)$$

subjected to the initial condition  $x_{21}^*(s)|_{s=0} = 1/\sqrt{\hbar}$ . By definition Eq. (50), the function  $\bar{x}_{21}(t, t')$  is only dependent on the time difference  $s = t - t'$ . Taking the operation of complex conjugation on both sides of Eq. (52) leads to the equation which is the same as Eq. (59). Besides,  $\bar{x}_{21}(s)|_{s=0} = 1/\sqrt{\hbar} = x_{21}^*(s)|_{s=0}$ . Therefore,  $x_{21}^*(t, t') = \bar{x}_{21}(t, t')$  does hold.

By definition Eq. (51) and with the help of Eqs. (52) and (53), we find that  $\bar{y}(t, t')$  sat-

isfies

$$\frac{\partial}{\partial t'} \bar{y}(t, t') = -i\omega_0 \bar{y}(t, t') - \int_0^{t'} dt_1 \alpha_1^*(t' - t_1) \bar{y}(t, t_1) + \int_0^t dt_1 \alpha_2^*(t' - t_1) \bar{x}_{21}(t, t_1). \quad (60)$$

The same equation can be obtained for  $y^*(t, t')$  from Eqs. (38) and (39). Moreover,  $\bar{y}(t, t')|_{t'=t} = y^*(t, t')|_{t'=t} = 1/\sqrt{\hbar}$ . Therefore, one proves  $\bar{y}(t, t') = y^*(t, t')$  and as a result,  $A_3(t) = B_3(t)$ . We have therefore demonstrated that the master equation Eq. (41) resulting from stochastic description is identical with that derived with influence functional method [24].

Some remarks on the calculation of the coefficients of the master equation are in order. As discussed above, our procedure provides a set of integral equations, while Xiong *et al.* [24] introduce an integro-differential equation or the equation of the related Green's function. It is straightforward to numerically independent both of the two schemes to determine the coefficients. Although we prove that these two frameworks give the identical results, their numerical performance might be different. Because the computational scaling for solving the integral equation is less favorable than solving the corresponding differential equation, the Green's method is preferred in practice.

## V. DRIVEN CAVITY DYNAMICS

Let us consider the cavity dynamics in the presence of a time-dependent external field  $\epsilon(t)$ . Now the Hamiltonian of the system reads  $\hat{H}_s(t) = \hbar\omega_0 a^\dagger a + \epsilon(t) (a + a^\dagger)$ . The master equation can be derived along the same line discussed in Sec. III. Although the external field only directly acts on the cavity system, and does not change the bath-induced stochastic fields, it does interfere with the bath during the evolution of the system. This effect is reflected in the change of dissipative operators. Starting with Eqs. (24)–(27), we solve the

required propagators and take the stochastic averaging to obtain

$$\begin{aligned}
\hat{O}_{s,11}(t, t') &= \frac{1}{\sqrt{\hbar}} \{a^\dagger, \tilde{\rho}_s(t)\} e^{-i\omega_0(t-t')} + \int_{t'}^t dt_1 \int_0^{t_1} dt_2 e^{-i\omega_0(t_1-t')} \alpha_1(t_1 - t_2) \hat{O}_{s,11}(t, t_2) \\
&\quad - \int_{t'}^t dt_1 \int_0^t dt_2 e^{-i\omega_0(t_1-t')} \alpha_2(t_1 - t_2) \hat{O}_{s,12}(t, t_2) - \frac{2i}{\hbar\sqrt{\hbar}} \int_{t'}^t dt_1 e^{-i\omega_0(t_1-t')} \epsilon(t_1) \tilde{\rho}_s(t), \\
\hat{O}_{s,12}(t, t') &= \frac{1}{\sqrt{\hbar}} [a^\dagger, \tilde{\rho}_s(t)] e^{-i\omega_0(t-t')} - \int_{t'}^t dt_1 \int_{t_1}^t dt_2 e^{-i\omega_0(t_1-t')} \alpha_1^*(t_2 - t_1) \hat{O}_{s,12}(t, t_2), \\
\hat{O}_{s,21}(t, t') &= \frac{1}{\sqrt{\hbar}} \{a, \tilde{\rho}_s(t)\} e^{i\omega_0(t-t')} + \int_{t'}^t dt_1 \int_0^{t_1} dt_2 e^{i\omega_0(t_1-t')} \alpha_1^*(t_1 - t_2) \hat{O}_{s,21}(t, t_2) \\
&\quad + \int_{t'}^t dt_1 \int_0^t dt_2 e^{i\omega_0(t_1-t')} \alpha_2(t_2 - t_1) \hat{O}_{s,22}(t, t_2) + \frac{2i}{\hbar\sqrt{\hbar}} \int_{t'}^t dt_1 e^{i\omega_0(t_1-t')} \epsilon(t_1) \tilde{\rho}_s(t), \\
\hat{O}_{s,22}(t, t') &= \frac{1}{\sqrt{\hbar}} [a, \tilde{\rho}_s(t)] e^{i\omega_0(t-t')} - \int_{t'}^t dt_1 \int_{t_1}^t dt_2 e^{i\omega_0(t_1-t')} \alpha_1(t_2 - t_1) \hat{O}_{s,22}(t, t_2).
\end{aligned}$$

We use the same reasoning as that in Sec. III to obtain

$$\begin{aligned}
\hat{O}_{s,11}(t, t') &= x_{11}(t, t') \{a^\dagger, \tilde{\rho}_s(t)\} + x_{12}(t, t') [a^\dagger, \tilde{\rho}_s(t)] + x_{13}(t, t') \tilde{\rho}_s(t), \\
\hat{O}_{s,12}(t, t') &= x_{21}(t, t') [a^\dagger, \tilde{\rho}_s(t)], \\
\hat{O}_{s,21}(t, t') &= x_{11}^*(t, t') \{a, \tilde{\rho}_s(t)\} - x_{12}^*(t, t') [a, \tilde{\rho}_s(t)] + x_{13}^*(t, t') \tilde{\rho}_s(t), \\
\hat{O}_{s,22}(t, t') &= x_{21}^*(t, t') [a, \tilde{\rho}_s(t)],
\end{aligned}$$

where all coefficients except  $x_{13}(t, t')$  are the same as that of the undriven case [Eqs. (38)–(40)]. The additional new function is defined by

$$x_{13}(t, t') = -\frac{2i}{\hbar\sqrt{\hbar}} \int_{t'}^t dt_1 e^{-i\omega_0(t_1-t')} \epsilon(t_1) + \int_{t'}^t dt_1 \int_0^{t_1} dt_2 e^{-i\omega_0(t_1-t')} \alpha_1(t_1 - t_2) x_{13}(t, t_2),$$

which is linearly dependent on the external driving field  $\epsilon(t)$ .

With these expressions the master equation now reads

$$\begin{aligned}
\frac{d\tilde{\rho}_s(t)}{dt} &= [-iA_1(t)a^\dagger a + C(t)a + D(t)a^\dagger, \tilde{\rho}_s(t)] + A_2(t) [2a\tilde{\rho}_s(t)a^\dagger - a^\dagger a\tilde{\rho}_s(t) - \tilde{\rho}_s(t)a^\dagger a] \\
&\quad + A_3(t) [a^\dagger \tilde{\rho}_s(t)a + a\tilde{\rho}_s(t)a^\dagger - a^\dagger a\tilde{\rho}_s(t) - \tilde{\rho}_s(t)aa^\dagger],
\end{aligned} \tag{61}$$

where

$$C(t) = -\frac{i}{\hbar} \epsilon(t) + \frac{\sqrt{\hbar}}{2} \int_0^t dt' \alpha_1(t - t') x_{13}(t, t')$$

and

$$D(t) = -\frac{i}{\hbar} \epsilon(t) - \frac{\sqrt{\hbar}}{2} \int_0^t dt' \alpha_1^*(t - t') x_{13}^*(t, t').$$

Here, the coefficients  $A_j(t)$  ( $j = 1 - 3$ ) are the same as that of the undriven case, which satisfy Eqs. (42)–(44). It becomes clear that there are effects of the external field on the system, one is the direct interaction and the other results in the very interplay between the driving field and dissipation. As a consequence, the external field can be applied to *control* dissipation, or via versa, dissipation can be used to modulate the external field.

## VI. MASTER EQUATION FOR TWO-STATE ATOMS IN VACUUM

The spontaneous decay of a two-state atom coupled to a vacuum is described by the Hamiltonian Eq. (1) with  $\hat{H}_s = -\hbar\omega_0\sigma_z/2$ ,  $\hat{f}_1 = \sigma^-$ , and  $\hat{f}_2 = \sigma^+$ , where  $\sigma_z$  is the pauli matrix, and  $\sigma^+$  and  $\sigma^-$  are the raising and lowering operators. They satisfy the commutation relations  $[\sigma^+, \sigma^-] = \sigma_z$ ,  $[\sigma^+, \sigma_z] = -2\sigma^+$ , and  $[\sigma^-, \sigma_z] = 2\sigma^-$ . This damped two-state model might provide fundamental understanding of decoherence and other features of the dynamics of a qubit coupled to a heat bath. It is no wonder that its master equation has been derived and explored by several authors with diversified theoretical methods. For instance, Garraway developed a pseudomode technique to solve the dynamics [25]. Through the solution of the Schrödinger equation for the entire system, Breuer and coworkers worked out the reduced density matrix and thereby proposed a simple derivation of the corresponding master equation by a brute force calculation of the derivative with respect to time [26, 27]. They also developed a stochastic wave function approach to simulate the dynamics [28]. Strunz *et al.* proposed a different stochastic Schrödinger function method to solve the dissipative dynamics of the model [29]. The reduced density matrix resulting from the Schrödinger equation was also exploited by Vacchini and coworkers who recently showed how to generate the exact master equations corresponding to the time-convolutionless form and to the Nakajima-Zwanzig non-Markovian form [30]. In the first paper on the stochastic description of quantum dissipative systems, one of the authors also demonstrated how to derive the master equation from the stochastic equation of motion [18]. His method is based on self-consistency of an ansatz related to a stochastic average and the derivation was not expounded in the paper [18].

It seems that all the derivations in the literature are not direct and straightforward within one theoretical framework. We will show the stochastic description does offer a good pass to the master equation from the equation for the random density matrix for

the system. Because the bath is the vacuum field, the temperature is zero. As a result,  $\coth[\hbar\omega/(2k_B T)] = 1$  and the response functions derived by Eqs. (13) and (14) become identical,  $\alpha_1(t) = \alpha_2(t) \equiv \alpha(t)$ . Therefore, the bath-induced stochastic fields determined by Eqs. (11) and (12) become

$$\bar{g}_1(t) = \frac{i\hbar\sqrt{\hbar}}{2} \int_0^t dt' \alpha(t-t') [-i\nu_{12}(t') + \nu_{22}(t') + i\nu_{32}(t') - \nu_{42}(t')] \quad (62)$$

and

$$\bar{g}_2(t) = -\frac{i\hbar\sqrt{\hbar}}{2} \int_0^t dt' \alpha^*(t-t') [i\nu_{11}(t') + \nu_{21}(t') + i\nu_{31}(t') + \nu_{41}(t')]. \quad (63)$$

The formal solution of the random density matrix of the system is still given by Eq. (21) where the forward and backward propagators  $U_1(t, 0)$  and  $U_2(0, t)$  are ruled by the corresponding Hamiltonians Eqs. (22) and (23) with  $\hat{f}_1 = \sigma^-$  and  $\hat{f}_2 = \sigma^+$ . Apparently, there are six complex Gaussian fields,  $\bar{g}_1(t)$ ,  $\bar{g}_2(t)$ ,  $\eta_{11}^*(t)$ ,  $\eta_{12}^*(t)$ ,  $\eta_{21}^*(t)$ , and  $\eta_{22}^*(t)$  involving in the dynamics. Note that all of the six Gaussian noises have zero means and null autocovariances. The average of a stochastic process generated, therefore, is fully determined by their non-vanishing cross-covariances. Given  $\bar{g}_1(t)$  and  $\bar{g}_2(t)$  by Eqs. (62) and (63), however, one can readily check that the white noises  $\eta_{12}^*(t)$  and  $\eta_{21}^*(t)$  are not correlated with other four and do not have any influence on the averaged dynamics. Therefore,  $\eta_{12}^*(t)$  and  $\eta_{21}^*(t)$  can be safely omitted when calculating the reduced density matrix.

To derive the master equation, we insert  $\bar{g}_1(t)$  and  $\bar{g}_2(t)$  into Eq. (2) and take stochastic averaging to obtain

$$\begin{aligned} i\hbar \frac{\partial \tilde{\rho}_s(t)}{\partial t} = & \left[ \hat{H}_s, \tilde{\rho}_s(t) \right] + \frac{i\hbar\sqrt{\hbar}}{2} \left[ \sigma^-, \int_0^t dt' \alpha(t-t') \hat{O}_{s,1}(t, t') \right] \\ & - \frac{i\hbar\sqrt{\hbar}}{2} \left[ \sigma^+, \int_0^t dt' \alpha^*(t-t') \hat{O}_{s,2}(t, t') \right], \end{aligned} \quad (64)$$

where the dissipative operators are

$$\begin{aligned} \hat{O}_{s,1}(t, t') = & M \left\{ -i \frac{\delta \rho_s(t)}{\delta \nu_{12}(t')} + \frac{\delta \rho_s(t)}{\delta \nu_{22}(t')} + i \frac{\delta \rho_s(t)}{\delta \nu_{32}(t')} - \frac{\delta \rho_s(t)}{\delta \nu_{42}(t')} \right\} \\ = & \frac{2}{\sqrt{\hbar}} M \{ \rho_s(t) \sigma_2^+(t, t') \} \end{aligned} \quad (65)$$

and

$$\begin{aligned} \hat{O}_{s,2}(t, t') = & M \left\{ i \frac{\delta \rho_s(t)}{\delta \nu_{11}(t')} + \frac{\delta \rho_s(t)}{\delta \nu_{21}(t')} + i \frac{\delta \rho_s(t)}{\delta \nu_{31}(t')} + \frac{\delta \rho_s(t)}{\delta \nu_{41}(t')} \right\} \\ = & \frac{2}{\sqrt{\hbar}} M \{ \sigma_1^-(t, t') \rho_s(t) \}, \end{aligned} \quad (66)$$

with  $\sigma_{1,2}^{\pm}(t, t') = U_{1,2}(t, t')\sigma^{\pm}U_{1,2}(t', t)$ . We like to stress that the derivation up to now is parallel to that illuminated in Sec. III. Now we need to find the explicit expressions of  $M\{\rho_s(t)\sigma_2^+(t, t')\}$  and  $M\{\sigma_1^-(t, t')\rho_s(t)\}$  in terms of  $\tilde{\rho}_s(t)$  and other known operators of the system. To this end, we consider their derivatives with respect to  $t'$ ,

$$\begin{aligned} \frac{\partial}{\partial t'} M\{\rho_s(t)\sigma_2^+(t, t')\} &= -i\omega_0 M\{\rho_s(t)\sigma_2^+(t, t')\} \\ &+ \int_0^{t'} dt_1 \alpha(t' - t_1) M\{\rho_s(t)\sigma_2^+(t, t_1)U_2(t, t')\sigma_z U_2(t', t)\} \end{aligned} \quad (67)$$

and

$$\begin{aligned} \frac{\partial}{\partial t'} M\{\sigma_1^-(t, t')\rho_s(t)\} &= i\omega_0 M\{\sigma_1^-(t, t')\rho_s(t)\} \\ &+ \int_0^{t'} dt_1 \alpha^*(t' - t_1) M\{U_1(t, t')\sigma_z U_1(t', t)\sigma_1^-(t, t_1)\rho_s(t)\}. \end{aligned} \quad (68)$$

By virtue of  $\sigma_z = 2\sigma^+\sigma^- - \text{I}$ , the two equations can be converted to

$$\begin{aligned} \frac{\partial}{\partial t'} M\{\rho_s(t)\sigma_2^+(t, t')\} &= -i\omega_0 M\{\rho_s(t)\sigma_2^+(t, t')\} + 2 \int_0^{t'} dt_1 \alpha(t' - t_1) M\{\hat{X}_1(t, t_1, t')\} \\ &- \int_0^{t'} dt_1 \alpha(t' - t_1) M\{\rho_s(t)\sigma_2^+(t, t_1)\} \end{aligned} \quad (69)$$

and

$$\begin{aligned} \frac{\partial}{\partial t'} M\{\sigma_1^-(t, t')\rho_s(t)\} &= i\omega_0 M\{\sigma_1^-(t, t')\rho_s(t)\} + 2 \int_0^{t'} dt_1 \alpha^*(t' - t_1) M\{\hat{X}_2(t, t_1, t')\} \\ &- \int_0^{t'} dt_1 \alpha^*(t' - t_1) M\{\sigma_1^-(t, t_1)\rho_s(t)\}, \end{aligned} \quad (70)$$

where

$$\hat{X}_1(t, t_1, t') = \rho_s(t)\sigma_2^+(t, t_1)\bar{\sigma}_2(t, t')$$

and

$$\hat{X}_2(t, t_1, t') = \bar{\sigma}_1(t, t')\sigma_1^-(t, t_1)\rho_s(t),$$

with  $\bar{\sigma}_{1,2}(t, t') = U_{1,2}(t, t')\sigma^{\pm}U_{1,2}(t', t)$ .

To find closed equations for  $M\{\rho_s(t)\sigma_2^+(t, t')\}$  and  $M\{\sigma_1^-(t, t')\rho_s(t)\}$ , therefore, we should evaluate  $M\{\hat{X}_1(t, t_1, t')\}$  and  $M\{\hat{X}_2(t, t_1, t')\}$ . When the first argument  $t$  is fixed,  $M\{\hat{X}_1(t, t_1, t')\}$  and  $M\{\hat{X}_2(t, t_1, t')\}$  can be taken as the functions of  $t_1$  and  $t'$ . For brevity,

the argument  $t$  for functions  $\hat{X}_1$  and  $\hat{X}_2$  will not be written. On taking their derivatives with respect to  $t_1$ , we obtain

$$\frac{\partial}{\partial t_1} M \left\{ \hat{X}_1(t_1, t') \right\} = -i\omega_0 M \left\{ \hat{X}_1(t_1, t') \right\} + \int_0^{t_1} dt_2 \alpha(t_1 - t_2) M \left\{ 2\hat{X}_1(t_2, t_1) \bar{\sigma}_2(t, t') - \hat{X}_1(t_2, t') \right\} \quad (71)$$

and

$$\frac{\partial}{\partial t_1} M \left\{ \hat{X}_2(t_1, t') \right\} = i\omega_0 M \left\{ \hat{X}_2(t_1, t') \right\} + \int_0^{t_1} dt_2 \alpha^*(t_1 - t_2) M \left\{ 2\bar{\sigma}_1(t, t') \hat{X}_2(t_2, t_1) - \hat{X}_2(t_2, t') \right\}. \quad (72)$$

We like to point out that the solutions for  $\hat{X}_1(t_1, t')$  and  $\hat{X}_2(t_1, t')$  can be many as long as their *stochastic averages* satisfy Eqs. (71) and (72). Because any solutions are sufficient for our purpose, we only consider the simple ones determined by

$$\frac{\partial}{\partial t_1} \hat{X}_1(t_1, t') = -i\omega_0 \hat{X}_1(t_1, t') + \int_0^{t_1} dt_2 \alpha(t_1 - t_2) \left[ 2\hat{X}_1(t_2, t_1) \bar{\sigma}_2(t, t') - \hat{X}_1(t_2, t') \right] \quad (73)$$

and

$$\frac{\partial}{\partial t_1} \hat{X}_2(t_1, t') = i\omega_0 \hat{X}_2(t_1, t') + \int_0^{t_1} dt_2 \alpha^*(t_1 - t_2) \left[ 2\bar{\sigma}_1(t, t') \hat{X}_2(t_2, t_1) - \hat{X}_2(t_2, t') \right], \quad (74)$$

with the initial conditions  $\hat{X}_1(t_1, t')|_{t_1=t'} = 0$  and  $\hat{X}_2(t_1, t')|_{t_1=t'} = 0$ .

As a result, we obtain  $\hat{X}_1(t_1, t') = 0$  and  $\hat{X}_2(t_1, t') = 0$ . Then Eqs. (69) and (70) become

$$\frac{\partial}{\partial t'} M \left\{ \rho_s(t) \sigma_2^+(t, t') \right\} = -i\omega_0 M \left\{ \rho_s(t) \sigma_2^+(t, t') \right\} - \int_0^{t'} dt_1 \alpha(t' - t_1) M \left\{ \rho_s(t) \sigma_2^+(t, t_1) \right\} \quad (75)$$

and

$$\frac{\partial}{\partial t'} M \left\{ \sigma_1^-(t, t') \rho_s(t) \right\} = i\omega_0 M \left\{ \sigma_1^-(t, t') \rho_s(t) \right\} - \int_0^{t'} dt_1 \alpha^*(t' - t_1) M \left\{ \sigma_1^-(t, t_1) \rho_s(t) \right\}. \quad (76)$$

They are integrated over time  $t'$ , namely,

$$M \left\{ \rho_s(t) \sigma_2^+(t, t') \right\} = e^{-i\omega_0(t'-t)} \tilde{\rho}_s(t) \sigma^+ + \int_{t'}^t dt_1 \int_0^{t_1} dt_2 e^{-i\omega_0(t'-t_1)} \alpha(t_1 - t_2) M \left\{ \rho_s(t) \sigma_2^+(t, t_2) \right\} \quad (77)$$

and

$$M \left\{ \sigma_1^-(t, t') \rho_s(t) \right\} = e^{i\omega_0(t'-t)} \sigma^- \tilde{\rho}_s(t) + \int_{t'}^t dt_1 \int_0^{t_1} dt_2 e^{i\omega_0(t'-t_1)} \alpha^*(t_1 - t_2) M \left\{ \sigma_1^-(t, t_2) \rho_s(t) \right\}. \quad (78)$$

Note that  $M \{ \rho_s(t) \sigma_2^+(t, t') \}^\dagger = M \{ \sigma_1^-(t, t') \rho_s(t) \}$ . By iteration, one can find that  $M \{ \rho_s(t) \sigma_2^+(t, t') \}$  and  $M \{ \sigma_1^-(t, t') \rho_s(t) \}$  possess the following forms,

$$M \{ \rho_s(t) \sigma_2^+(t, t') \} = x(t, t') \tilde{\rho}_s(t) \sigma^+ \quad (79)$$

and

$$M \{ \sigma_1^-(t, t') \rho_s(t) \} = x^*(t, t') \sigma^- \tilde{\rho}_s(t). \quad (80)$$

Because the operators  $\tilde{\rho}_s(t) \sigma^+$  and  $\sigma^- \tilde{\rho}_s(t)$  are arbitrary, the coefficient  $x(t, t')$  is determined by Eq. (77), which obeys

$$x(t, t') = e^{-i\omega_0(t'-t)} + \int_{t'}^t dt_1 \int_0^{t_1} dt_2 e^{-i\omega_0(t'-t_1)} \alpha(t_1 - t_2) x(t, t_2). \quad (81)$$

With the explicit expressions of  $M \{ \rho_s(t) \sigma_2^+(t, t') \}$  and  $M \{ \sigma_1^-(t, t') \rho_s(t) \}$ , Eq. (64) immediately becomes the resulting master equation. For the spontaneous decay of a two-state atom it reads

$$\begin{aligned} \frac{d\tilde{\rho}_s(t)}{dt} = & -\frac{i}{\hbar} [\hat{H}_s, \tilde{\rho}_s(t)] - i \frac{S(t)}{2} [\sigma^+ \sigma^-, \tilde{\rho}_s(t)] \\ & + R(t) \left[ \sigma^- \tilde{\rho}_s(t) \sigma^+ - \frac{1}{2} \sigma^+ \sigma^- \tilde{\rho}_s(t) - \frac{1}{2} \tilde{\rho}_s(t) \sigma^+ \sigma^- \right], \end{aligned} \quad (82)$$

where  $S(t)$  and  $R(t)$  are the time-dependent coefficients for the descriptions of a frequency shift and a decay rate, respectively. Their expressions are

$$R(t) = 2\text{Re} \left[ \int_0^t dt' \alpha^*(t - t') x^*(t, t') \right]$$

and

$$S(t) = 2\text{Im} \left[ \int_0^t dt' \alpha^*(t - t') x^*(t, t') \right].$$

## VII. CONCLUSION

The main goal of investigating dissipative systems is to solve their properties, in particular, to reveal the dissipative effect on their dynamics or Brownian motion. From the system plus environment model, we have shown [18] that the coupling to the environment can be rigorously mapped into stochastic fields and thereby provided a microscopic description of the Brownian motion. The resulting equation of motion for the density operator is

a stochastic Liouville equation and the statistical average of the solution gives the reduced density matrix, the key quantity defining the system. Like the classical counterpart, the Langevin equation, the stochastic Liouville equation offers a convenient way for the numerical simulation of quantum Brownian dynamics, however, its efficiency is seriously limited due to the slow convergence of stochastic averaging [19, 21, 22]. It is therefore desirable to derive the equation of motion for the reduced density operator or the master equation if it exists, given the stochastic Liouville equation. A general procedure was suggested in [18] and the detailed derivation of the master equation for the dissipative harmonic oscillator was presented in [23]. This paper tackles the dissipative dynamics of quantum optics in the same light.

We first worked out the bath-induced stochastic fields comprising two terms with the rotating-wave-approximation type interaction and then showed how to determine the “dissipation operators” for a single cavity mode. Similar to the case of the dissipative harmonic oscillator described by the Caldeira-Leggett model, the coefficients of the master equation for single cavity mode are determined by a set of integral equations. It is shown that our result is identical to that derived by virtue of path integral technique [24]. The master equation of a dissipative cavity mode at a driving field was also derived and the display between the dissipation and the driving field was pointed out. To show that the stochastic formulation is a systematic method for treating dissipative dynamics in quantum optics, we finally explained how to acquire the master equation for the spontaneous decay of two-state atoms coupled to the vacuum field. For solving the master equation, because the integral equation is time-nonlocal, it would be better to transform it into a differential one for a favorable numerical implementation, if such a transformation is available.

There are still many interesting questions in the stochastic formulation of dissipation. A related one to the derivation of the master equation is for what kinds of system and couplings such an equation exists. Notwithstanding, as the quantum dissipation becomes an important and subtle issue and attracts more and more attention in the community of quantum optics and quantum information, it is expected that the stochastic description will be a powerful tool in either theoretical analysis or numerical simulations.

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- [1] A. Einstein, *Ann. Phys.* **17**, 549 (1905).
- [2] P. Langevin, *C. R. Acad. Sci.(Paris)*, **146**, 530 (1908).
- [3] U. Weiss, *Quantum Dissipative Systems*, 3rd ed. (World Scientific, Singapore, 2008).
- [4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
- [5] G. W. Ford, J. T. Lewis, and R. F. O'Connell, *Phys. Rev. A* **64**, 032101 (2001).
- [6] M. Schlosshauer, *Rev. Mod. Phys.* **76**, 1267 (2005).
- [7] W. H. Zurek, *Phys. Today* **44**, 36 (1991); *Rev. Mod. Phys.* **75**, 715 (2003).
- [8] A. Buchleitner, C. Viviescas, and M. Tiersch, *Entanglement and Decoherence* (Springer-Verlag, Berlin, 2009).
- [9] R. Kubo, *J. Math. Phys.* **4**, 174 (1963).
- [10] S. Nakajima, *Prog. Theor. Phys.* **20**, 948 (1958).
- [11] R. Zwanzig, *J. Chem. Phys.* **33**, 1338 (1960).
- [12] R. P. Feynman and F. L. Vernon, *Ann. Phys.* **24**, 118 (1963).
- [13] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
- [14] A. O. Caldeira and A. J. Leggett, *Physica A* **121**, 587 (1983); **130**, 374(E) (1985); *Ann. Phys.* **149**, 374 (1983); **153**, 445(E) (1984).
- [15] R. Benguria and M. Kac, *Phys. Rev. Lett.* **46**, 1 (1981).
- [16] G. W. Ford and M. Kac, *J. Stat. Phys.* **46**, 803 (1987); G. W. Ford, J. T. Lewis, and R. F. O'Connell, *Phys. Rev. A* **37**, 4419 (1988).
- [17] H. Risken, *The Fokker-Planck Equation*, 2nd ed. (Springer-Verlag, Berlin, 1989).
- [18] J. Shao, *J. Chem. Phys.* **120**, 5053 (2004); *Chem. Phys.* **322**, 187 (2006); **370**, 29 (2010).
- [19] Y. A. Yan, F. Yang, Y. Liu, and J. Shao, *Chem. Phys. Lett.* **395**, 216 (2004).

- [20] Y. Tanimura and R. Kubo, J. Phys. Soc. Jpn. **58**, 101 (1989); Y. Tanimura, *ibid.* **75**, 082001 (2006).
- [21] Y. Zhou, Y. Yan, and J. Shao, Europhys. Lett. **72**, 334 (2005).
- [22] Y. Zhou and J. Shao, J. Chem. Phys. **128**, 034106 (2008).
- [23] H. Li, S. Wang, and J. Shao, Phys. Rev. E **84**, 051112 (2011).
- [24] H. N. Xiong, W. M. Zhang, X. G. Wang, and M. H. Wu, Phys. Rev. A **82**, 012105 (2010).
- [25] B. M. Garraway, Phys. Rev. A **55**, 2290 (1997).
- [26] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University, Oxford, 2002).
- [27] H.-P. Breuer, B. Kappler, and F. Petruccione, Phys. Rev. A **59**, 1633 (1999).
- [28] B. Vacchini and H.-P. Breuer, Phys. Rev. A **81**, 042103 (2010).
- [29] W. T. Strunz, L. Diósi, and N. Gisin, Phys. Rev. Lett. **82**, 1801 (1999).
- [30] A. Smirne and B. Vacchini, Phys. Rev. A **82**, 022110 (2010).
- [31] H. Carmichael, *An Open Systems Approach to Quantum Optics*, Lecture Notes in Physics, Vol. 18 (Springer-Verlag, Berlin, 1993).
- [32] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University, Cambridge, 1995).
- [33] R. J. Glauber, *Quantum Theory of Optical Coherence* (Wiley-VCH, Weinheim, 2007).
- [34] C. W. Gardiner and P. Zoller, *Quantum Noise*, 2nd ed. (Springer-Verlag, Berlin, 2000).
- [35] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, Berlin, 1995).
- [36] P. Meystre and M. Sargent, *Elements of Quantum Optics*, 4th ed. (Springer-Verlag, Berlin, 2007).
- [37] C. H. Fleming, N. I. Cummings, C. Anastopoulos, and B. L. Hu, J. Phys. A: Math. Theor. **43**, 405304, (2010).
- [38] H. T. Tan and W. M. Zhang, Phys. Rev. A **83**, 032102 (2011).
- [39] W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).
- [40] W. Dittrich and M. Reuter, *Classical and Quantum Dynamics*, 3rd ed. (Springer-Verlag, Berlin, 2001).
- [41] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, 5th ed. (World Scientific, Singapore, 2009).
- [42] R. P. Feynman, *Statistical Mechanics* (Benjamin, New York, 1972).

[43] E. A. Novikov, Sov. Phys. JETP **20**, 1290 (1965).